Journal of Computational and Applied Mathematics 16 (1986) 135–143
North-Holland

On selecting the best contender *

Young Jack LEE
Biometry and Field Studies Branch, National Institute of Neurological and Communicative Disorders and Stroke, Bethesda, MD 20205, U.S.A.

Edward J. DUDEWICZ
Department of Mathematics, Syracuse University, Syracuse, NY 13210, U.S.A.

Received 12 December 1984
Revised 7 August 1985

Abstract: When selecting the best contender based on relative rankings of k contenders, the definition of the best contender is not necessarily straightforward. In particular, if different summary statistics lead to selecting different best contenders, which summary statistic is to be relied upon is not usually clear. In this paper we define the best contender, discuss procedures compatible with the definitions, and compare the procedures via relative efficiency. An example data set is presented and discussed.

Keywords: Best contender, incomplete rank score matrix, selection procedure, runner-up selection, relative efficiency.

1. Introduction

The statistical problem we consider in this paper can be exemplified by the annual Heisman Trophy winner selection. Each year the sports writers and reporters rank college football players (3 for the best, 2 for the second best, 1 for the third best, and 0 for all others) at the end of each season, and the player with the highest total score is awarded the Heisman Trophy. Usually the winner not only receives the highest total score but also receives the most first-place votes. In some cases, however, the player with the highest total score is not the player with the most first-place votes, however the highest total score getter is nonetheless selected as the winner. Another example is: collegiate football teams are ranked at the end of the postseason bowl games (usually January 2) every year by sports writers and reporters (the AP poll), and a team is selected as the 'mythical champion'. The method used is that each judge gives 20 points for the best, 19 points for the 2nd best, ..., 1 point for the 20th best, and 0 points for all others, and that the team with the highest total score is dubbed as the champion of the year, the one with the 2nd highest total score the 2nd best team and so on. The champion title is most important, and the champion is not always the team with the most best-votes. Who is the champion? How do we define the champion and select it? As another example, in selecting for the 1976 Frank Wilcoxon prize for the best application paper in the Technometrics 1976 issues, nineteen referees ranked

* This investigation was supported by Grant Number 1 R01 CA26254, awarded by the National Cancer Institute, DHEW.

six finalists, not all the referees completing the ranking. Different summary statistics select the awardee differently (Lee and Dudewicz [6]).

The above examples fit into the n-judge-k-contender model where k contenders may or may not be completely ranked by n judges. In this paper we consider the problem of selecting a winner for a given partial rank matrix. To this end we will consider the definition of the best contender in the n-judge-k-contestant model, and discuss procedures for selecting the best contender consistent with the definition.

2. Definitions, procedures and notations

Suppose that the number of contenders is k ($k = 2$ or 3 or 4 or ...) and that each expert ranks at least t ($1 \leqslant t \leqslant k$) contenders from best to worst. Denote the ith contender by C_i.

Let ϕ_{il} denote the probability that contender C_i is rated as the $(k-l+1)$st best, among $C_1, \ldots, C_k$, by any expert. Define

$$\mu_i(t) = \sum_{l=k-t+1}^{k} (l+t-k)\phi_{il}.$$

Call the contender associated with $\max(\mu_1(t), \ldots, \mu_k(t))$ the *best contender* (by ranking) *t-out-of-k*, and denote that contender as C-best(t). It is, of course, possible that different contenders are C-best(t) for different values of t, a fact which is related to the so-called 'voting paradox' of Arrow's Impossibility Theorem [1,2]. More precisely, a method of combining votes is called *Pareto* if and only if for any pair of contenders x and y, such that all referees strictly prefer x to y, the method selects x over y. Also, a method of combining votes is called *independent of irrelevant alternatives* if which of two alternatives x and y is selected is not affected by attitudes (preferences) towards any third alternative. Arrow's Impossibility Theorem states that only the *dictatioral method* (which always selects the candidate preferred by a stated particular referee) is both Pareto and independent of irrelevant alternatives when the number of contenders k is greater than 2. Hence we should not be surprised if different values of t lead to different selections by our procedures below: while the methods for each t are Pareto, they cannot be independent of irrelevant alternatives by the Impossibility Theorem. However, this does not mean that we will always obtain inconsistent selections, since often not all preference orderings which are mathematically possible will occur in significant numbers.

The selection procedure $R(t)$ for finding the C-best(t) contender is: Assign scores $t, t-1, \ldots, 2$, 1 and $k-t$ 0's to the k contenders, using the rank score matrix of the n judges. (The higher the score, the better a contender is rated.) Denote by $R_{ji}(t)$ the score assigned to C_i by the jth referee. Let $V_i(t) = \sum_{j=1}^{n}(t) R_{ji}(t)$, and select the contender with the largest of $V_1(t), \ldots, V_k(t)$

$$\max(V_1(t), V_2(t), \ldots, V_k(t)),$$

as the best. If two or more tie either split the first place or select one based on other considerations (or break the tie by randomization). The first two options are recommended for practice. (The last option is required for theoretical development of selection procedures below.)

Let $\mu_{[1]}(t) \leqslant \mu_{[2]}(t) \leqslant \cdots \leqslant \mu_{[k]}(t)$ denote ordered unknown values of $\mu_1(t), \mu_2(t), \ldots, \mu_k(t)$ and $\phi_{(i)l}$ be the probability associated with $\mu_{[i]}(t)$. The contestant associated with $\mu_{[i]}(t)$ is unknown for each i and denoted by $C_{(i)}(t)$. Let $V_{(i)}(t)$ denote the statistic associated with $C_{(i)}(t)$. Call the following event correct selection (CS):

$$V_{(k)}(t) = \max(V_1(t), \ldots, V_k(t)).$$

3. Mathematical details; relative efficiency; choice of t

Below we compare efficiencies of $R(1), R(2), \ldots, R(k)$ in selecting the best contender. This comparison makes sense only if the best contender is not definition-specific, namely $C_{(k)}(1) = \cdots = C_{(k)}(k)$. This definition-nonspecificity is usually acceptable in the parametric case: for example suppose that $(X_1, \ldots, X_k)$ are k independent random variables with distribution functions $F(X-\theta_1), \ldots, F(X-\theta_k)$ respectively, and that ϕ_{il} is the probability that X_i is the lth order statistic among $(X_1, \ldots, X_k)$. It is, in general, true that $\max(\mu_1(1), \ldots, \mu_k(1))$ is associated with $\max(\theta_1, \ldots, \theta_k)$ [5], and it is conjectured to be true for most F's that $\max(\mu_1(t), \ldots, \mu_k(t))$ is associated with $\max(\theta_1, \ldots, \theta_k)$ for every t. In the case of the n-judge-k-contender model, however, it is possible that the best contender depends on the definition because of the voting paradox. In spite of the voting paradox, experiences (such as the Heisman trophy winner, the collegiate football champion, the annual *Technometrics* prize winners, etc.) indicate that the best contender in most cases does not vary with the choice of t. Therefore in the following development it is realistic to assume that the best contender is the same regardless of the choice of t.

The strength of a selection procedure is measured by the probability of correct selection. Denote the event of CS of the procedure $R(t)$ by $P(\mathrm{CS} \mid R(t))$. If $P(\mathrm{CS} \mid R(t)) \leqslant P(\mathrm{CS} \mid R(t'))$ when we have n referees and k contenders, then procedure $R(t')$ should be preferred.

For notational convenience only, and without loss of generality, we assume that the kth contestant is the best. Thus the statistic and parameters associated with C_k are simply $V_k(t)$ and ϕ_{kl}.

Let n, k, t, and λ^*, $1 < \lambda^* < \infty$, be fixed and suppose that

$$\mu_k(t) \geqslant \lambda^* \mu_i(t), \quad i = 1, \ldots, k-1. \tag{1}$$

The infimum of $P[\mathrm{CS} \mid \mu_k(t) \geqslant \lambda^* \mu_i(t), R(t)]$ is sought in the subspace of ϕ_{il}'s satisfying (1), because $\mu_i(t)$'s are defined on ϕ_{il}'s. Define the slipped configuration of ϕ as following:

$$\mathrm{SC}(\phi)\colon \phi_{kk} = \lambda^* \phi_{kl} = \lambda^* \phi_{l'k}, \quad \phi_{ll'} = \phi_{l'l}, \qquad 1 \leqslant l, \quad l' \leqslant k-1. \tag{2}$$

The $\mathrm{SC}(\phi)$ defined by (2) satisfies (1), and therefore is a point in the subspace of ϕ_{il}'s satisfying (1). Therefore for $t \geqslant 2$

$$\inf_{\mu_k(t) \geqslant \lambda^* \mu_i(t)} P[\mathrm{CS} \mid \mu_i(t), R(t)] \leqslant P[\mathrm{CS} \mid \mathrm{SC}(\phi), R(t)]. \tag{3}$$

For $t = 1$

$$\inf_{\mu_k(1) \geqslant \lambda^* \mu_i(1)} P[\mathrm{CS} \mid \mu_i(1), R(1)] = P[\mathrm{CS} \mid R(1), \phi_{kk} = \lambda^* \phi_{lk}, 1 \leqslant l \leqslant k-1]. \tag{4}$$

Note that (3) and (4) are more general than (3) and (4) in Lee and Dudewicz [6], because the space defined by (1) is a superset of the corresponding space in that earlier paper. Thus, the relative efficiency results (see equation (5) below) are more generally applicable. In fact

$P[\mathrm{CS} \mid \phi_{ij}, R(1)]$ is a function of only $\phi_{kk}, \phi_{(k-1)k}, \ldots, \phi_{2k}$, and ϕ_{1k}, and a proof of (4) is given by Kesten and Morse [3]. (Note that without the randomization the equality (4) is true only asymptotically, since as $n \to \infty$, the probability that randomization is required goes to zero.)

We can compare the selection procedures $R(t)$ by computing $P(\mathrm{CS} \mid R(t))$ for each t with fixed n, k, and λ^*. Instead of computing $P[\mathrm{CS} \mid R(t)]$ to compare $R(t)$'s for given n, k and λ^*, however, we approximate $P[\mathrm{CS} \mid R(t)]$ (see Appendix), equate the approximation to a given P^* $(1/k < P^* < 1)$, and solve the smallest n needed to satisfy the equation. Denote that n by $n_{k,t}(\lambda^*, P^*)$. The ratio $n_{k,t}(\lambda^*, P^*)/n_{k,t'}(\lambda^*, P^*)$, $1 \leqslant t \neq t' \leqslant k$, is called the *relative efficiency* of $R(t')$ with respect to $R(t)$, denoted by $\mathrm{Eff}[R(t'), R(t)]$. If $\mathrm{Eff}[R(t'), R(t)] \geqslant 1$, then procedure $R(t')$ is at least as efficient as $R(t)$. *Of particular interest is* $\mathrm{Eff}[R(1), R(t)]$. Since $\mathrm{Eff}[R(1), R(t)]$ requires a computation for each combination of (k, λ^*, P^*), we instead compute and obtain (see the Appendix)

$$\lim_{\lambda^* \to 1} \mathrm{Eff}[R(1), R(t)] = \frac{t+1}{3} \frac{(k-1)(4tk+2k-3t^2-3t)}{t(2k-t-1)^2}. \tag{5}$$

Table 1
Rank scores of 7 contenders for the 1976 Youden prize [a]

Referee	C_1	C_2	C_3	C_4	C_5	C_6	C_7
1	5	3	2	6	7	1	4
2	3	1	6	2	5	4	7
3	2	6	3	4	7	1	5
4	3	6	7	1	4	2	5
5	2	3	6	4	5	1	7
6	7	3	4	5	6	1	2
7	6	5	2	4	3	1	7
8	3	2	1	7	6	4	5
9	5	6	3	2	4	7	1
10	7	3	4	2	5	1	6
11	5	6	4	3	7	2	1
12	3	4	5	1	2	6	7
13	3	7	–	–	5	4	6
14	3	–	4	7	5	–	6
15	6	5	–	7	3	–	4
16	5	6	–	–	–	4	7
17	7	5	–	6	4	–	–
18	5	6	4	–	–	–	7
19	5	6	–	–	4	–	7
20	4	7	6	–	5	–	–
21	–	7	6	–	–	5	–
22	6	–	–	–	5	–	7
23	–	6	–	–	7	–	–
24	–	–	–	–	–	7	–

[a] The referee numbers are arranged for convenience in preparing the tables, Nonranked contenders' rank scores are not assigned. For example referee 13 ranked C_2 as the best, C_7 as the second best, C_5 as the third best, C_6 as the fourth best, C_1 as the fifth best and others nonranked. Therefore rank scores (7, 6, 5, 4, 3) were given to $(C_2, C_7, C_5, C_6, C_1)$, and C_3 and C_4 were not assigned rank scores.

Table 2
Rank scores of 6 finalists for the 1976 Wilcoxon prize [a]

Referee	C_1	C_2	C_3	C_4	C_5	C_6
1	6	2	3	4	1	5
2	2	4	5	3	1	6
3	5	2	4	3	6	1
4	5	6	2	4	3	1
5	2	3	4	1	6	5
6	3	4	2	5	1	6
7	5	1	4	3	2	6
8	2	6	3	5	4	1
9	5	4	3	2	1	6
10	6	5	1	2	4	3
11	4	6	–	3	5	–
12	6	4	5	–	–	3
13	–	–	4	6	3	5
14	3	5	–	6	4	–
15	3	4	–	6	5	–
16	–	4	3	6	–	5
17	5	–	4	6	–	–
18	–	4	6	5	–	–
19	–	4	–	5	–	6

[a] The referee numbers are arranged for convenience in preparing the tables. Rank scores were allocated similarly as in Table 1.

If $t = k$,

$$\lim_{\lambda^* \to 1} \mathrm{Eff}[R(1),\ R(k)] = \tfrac{1}{3}(k+1).$$

This efficiency result is intuitive (as an anonymous referee explains, the additional detail of the rank matrix serves to obscure the process of selecting the best contender).

4. Example: the Wilcoxon and Youden prizes

Each year, the Chemical Division of the American Society for Quality Control awards the Frank Wilcoxon Prize and the Jack Youden Prize. These Prizes are awarded for articles in *Technometrics*, the Wilcoxon being for the best practical application paper and the Youden for the best expository paper. In the 1976 competition (held in 1977) for these prizes, all the articles appearing in the 1976 volume of *Technometrics* formed a pool of candidates and fifty referees selected, from the pool, six articles for the Wilcoxon award competition and seven articles for the

Table 3
Sum of all scores in Table 2 with missing scores replaced by average scores

C_1	C_2	C_3	C_4	C_5	C_6
69	71.5	59.5	76.5	55	67.5

Table 4
Frequency of the first place vote for the Wilcoxon prize

C_1	C_2	C_3	C_4	C_5	C_6
3	3	1	5	2	5

Table 5
Relative efficiency of $R(1)$ when $k=6$

t	2	3	4	5	6
$\lim_{\lambda^* \to 1} \mathrm{Eff}[R(1), R(t)]$	1.30	1.67	2.04	2.33	2.33

Youden award competition. Referees were asked to rank the final contenders from best to worst according to their judgment on the overall quality, originality, scientific contribution, and readability of each contender. Referees were not mandated to complete the ranking, and as a result we obtained the incomplete rank score matrices of Tables 1 and 2.

Selecting the Youden Prize winner did not pose any problem, because selection procedures $R(1),\ldots, R(7)$ chose the contender C_7 as the best. But different selection procedures resulted in different winners of the Wilcoxon Prize [6]. For example, according to the selection procedure $R(6)$, the contender C_4 should be awarded the Wilcoxon Prize (Table 3), while according to the selection procedure $R(1)$, the Wilcoxon Prize should be split between C_4 and C_6 (Table 4). The evaluation of $\mathrm{Eff}[R(1), R(t)]$ for $k=6$ shows that $R(1)$ is the most efficient procedure as demonstrated in Table 5 when selecting the best contender based on the relative rank matrix. (Winners of the Technometrics Prizes for 1969–1978 are listed on pages 588–589 of the November 1979 (Volume 21) issue of *Technometrics*.)

5. Concluding remarks

When selecting the best from k contenders based on relative ranks by n judges, the relative efficiency result shows that selecting the contender with most first place votes is most efficient. Note that, whether inequality (3) is strict or not, the conclusion obtained does not change. If inequality (3) is strict, then the efficiency given by (5) is a lower bound for $\lim_{\lambda^* \to 1} \mathrm{Eff}[R(1), R(t)]$. We conjecture, however, that inequality (3) is not strict under the assumption that the ϕ_{ij}'s behave well.

Even though a runner-up contender (for example, for honorable mention) is not selected for the Frank Wilcoxon and Jack Youden Prizes, we will now note 'runner-up selection procedures', because the need for such procedures does arise in other applications. (For example, if a 'winner' is offered a job by a manager, the manager wishes to have at least one next-in-line candidate in case the first declines the offer.) The procedure we propose is: select, as the runner-up, the contender who is rated best most frequency among the set of contenders (with the one selected as the best omitted) by referees who rank at least two from the top. The Jack Youden Prize rank score matrix can be used as an example. When the 1976 Jack Youden Prize winner C_7 is not considered, the frequency distribution of the highest scores is, as obtained from Table 1,

C_1	C_2	C_3	C_4	C_5	C_6
5	6	3	3	4	3

and we select C_2 as the runner-up. The overall characteristics of the procedure for selecting the best and runner-up contenders for the n-judge-k-contender model are an area meriting further study.

Finally we note that the relative efficiency result $\mathrm{Eff}[R(1), R(k)] = \frac{1}{3}(k+1)$ is not unique to the particular model studied, but is also observed in some parametric models [4].

Appendix

We derive an approximation to $P(\mathrm{CS} \mid R(t)]$ and equation (5). Let Γ be the collection of all n by k rank score matrices:

$$\Gamma = \left\{ r_n = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nk} \end{pmatrix} : \text{all possible rank scores} \right\}.$$

Each row $(r_{j1}, r_{j2}, \ldots, r_{jk})$ of r_n is a permutation of $(1, 2, \ldots, k)$. There are $(k-1)!$ possible arrangements of $(1, 2, \ldots, k)$ in each of which $r_{jk} = l$ $(1 \leqslant l \leqslant k)$. Hence ϕ_{kl} can be decomposed into $(k-1)!$ parts ϕ_{klu}, $\phi_{klu} \geqslant 0$ and $\Sigma_u \phi_{klu} = \phi_{kl}$. That is $P[(R_{11}, \ldots, R_{1k}) = (r_{11}, \ldots, r_{1(k-1)}, l)$, $r_{1i} \neq l]$ is one of the ϕ_{klu}'s. Since

$$\begin{aligned} \phi_{il} = \sum P\big[&(R_{11}, \ldots, R_{1(i-1)}, R_{1i}, R_{1(i+1)}, \ldots, R_{1k}) \\ &= (r_{11}, \ldots, r_{1(i-1)}, l, r_{1(i+1)}, \ldots, r_{1k})\big] \end{aligned}$$

where the sum is over all permutations of $(1, \ldots, l-1, l+1, \ldots, k)$ for $(r_{11}, \ldots, r_{1(i-1)}, r_{1(i+1)}, \ldots, r_{1k})$, ϕ_{il} is expressible as a sum of some $\phi_{kl'u}$'s, $l' \neq l$.

Let SC(ϕ) (slipped configuration) denote the configuration

$$\phi_{kk} = \lambda^* \phi_{kl} = \lambda^* \phi_{l'k}, \quad \phi_{ll'} = \phi_{l'l}, \qquad 1 \leqslant l, \quad l' \leqslant k.$$

Under the configuration SC(ϕ),

$$\begin{aligned} &\phi_{kl} = 1/(k + \lambda^* - 1), \quad 1 \leqslant l \leqslant k-1, \\ &\phi_{kk} = \lambda^*/(k + \lambda^* - 1), \quad \text{and} \quad \phi_{klu} = \phi_{kl}/(k-1)!. \end{aligned}$$

It can also be shown that under SC(ϕ) with given t

$$P[(R_{1l}, R_{1k}) = (s, r)] = \begin{cases} 0 & \text{if } s = r, \\ \phi_{k(k-t+r)}/(k-1) & \text{if } s \neq r, \end{cases}$$

and

$$\begin{aligned} &P[(R_{1l}, R_{1l'}) = (s, r), 1 \leqslant l, l' \leqslant k-1] \\ &\quad = \begin{cases} 0 & \text{if } s = r, \\ (1 - \phi_{k(k-t+r)} - \phi_{k(k-t+s)})/(k-1)(k-2) & \text{if } s \neq r. \end{cases} \end{aligned}$$

Denote $n^{-1/2}(V_k - V_i)$, $1 \leqslant i \leqslant k-1$, by D_i. Using the above relationship, after rather involved computations, we obtain for fixed t

$$E\{D_i \mid \mathrm{SC}(\phi)\} = \frac{n^{1/2} t(\lambda^* - 1)(2k - t - 1)}{2(k + \lambda^* - 1)(k-1)},$$

$$\operatorname{Var}\{D_i \mid \mathrm{SC}(\phi)\} = m_{2i} - m_{1i}^2 + \frac{k+1}{k-1}\left(m_{2k} - m_{1k}^2\right),$$

and

$$\begin{aligned}\operatorname{Cov}\{D_i, D_{i'} \mid \mathrm{SC}(\phi)\} = {} & \frac{k+1}{k-1}\left(m_{2k} - m_{1k}^2\right) + \frac{1}{(k-1)(k-2)} \\ & \times \left\{\frac{t^2(t+1)^2}{4} - \frac{t(t+1)(2t+1)}{6} - t(t+1)m_{1k} + 2m_{2k}\right\} \\ & - m_{1i}m_{1i'},\end{aligned}$$

where

$$m_{1k} = E\{V_k \mid \mathrm{SC}(\phi)\} = \frac{1}{2(k + \lambda^* - 1)}\{2\lambda^* + t - 1\},$$

$$m_{1i} = E\{V_i \mid \mathrm{SC}(\phi)\} = \frac{1}{k-1}\left\{\frac{t(t+1)}{2} - m_{1k}\right\},$$

$$m_{2k} = E\{V_k^2 \mid \mathrm{SC}(\phi)\} = \frac{t}{6(k + \lambda^* - 1)}\{2t^2 - 3t(1 - 2\lambda^*) + 1\},$$

and

$$m_{2i} = E\{V_i^2 \mid \mathrm{SC}(\phi)\} = \frac{t(t+1)(2t+1)}{6(k-1)} - \frac{m_{2k}}{k-1}.$$

For a large sample, the right-hand side of (3) (in the case of $t = 1$, that of (4)) can be approximated by

$$P(D_1 > 0,\ D_2 > 0, \ldots, D_{k-1} > 0). \tag{A.1}$$

Applying a multivariate central limit theorem to $(D_1, D_2, \ldots, D_{k-1})$, we obtain the following approximation to $P[(\mathrm{CS} \mid R(t)]$:

$$\int_{-\infty}^{\infty} \Phi^{k-1}\left(\frac{z}{a(\lambda^*)} + \frac{n^{1/2} t(\lambda^* - 1)(2k - t - 1) b(\lambda^*)}{2(k + \lambda^* - 1)(k-1)\tau(\lambda^*) a(\lambda^*)}\right) \mathrm{d}\Phi(z), \tag{A.2}$$

where

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} \exp(-x^2/2)\, \mathrm{d}x,$$

$$a(\lambda^*) = \left\{\frac{\tau^2(\lambda^*) - \gamma(\lambda^*)}{\gamma(\lambda^*)}\right\}^{1/2}, \qquad b(\lambda^*) = \left\{1 + a^2(\lambda^*)\right\}^{1/2},$$

$$\tau^2(\lambda^*) = \operatorname{Var}\left\{n^{-1/2}(V_k(t) - V_1(t))\right\},$$

and

$$\gamma(\lambda^*) = \operatorname{Cov}\left\{n^{-1/2}(V_k(t) - V_1(t)),\ n^{-1/2}(V_k(t) - V_2(t))\right\}.$$

Equation (A.2) can be computed numerically using Gaussian quadrature. Approximation by (A.2) is fairly reliable, and was off by less than 0.01 for the cases studied.

Since $a(\lambda^*) \to 1$ as $\lambda^* \to 1$, letting

$$\rho_t(\lambda^*) = \frac{n^{1/2} t(\lambda^* - 1)(2k - t - 1) b(\lambda^*)}{2(k + \lambda^* - 1)(k - 1)\tau(\lambda^*) a(\lambda^*)},$$

by directly comparing (A.2) with $t = 1$ to (A.2) with $t > 1$,

$$\lim_{\lambda^* \to 1} \mathrm{Eff}[R(1),\ R(t)] = \lim_{\lambda^* \to 1} \{\rho_1(\lambda^*)/\rho_t(\lambda^*)\}^2,$$

which results in (5) upon noting that

$$\lim_{\lambda^* \to 1} \tau^2(\lambda^*) = \frac{t(t+1)}{6k(k-1)}(4tk + 2k - 3t^2 - 3t).$$

(Another way of obtaining (5) is: let $(\lambda^* - 1) = O(n^{-1/2})$ and directly compare (A.2) with $t = 1$ to (A.2) with $t > 1$ in the limit of n; this method is more conventional. Both, however, give the same result.)

References

[1] K.J. Arrow, Rational choice functions and orderings, *Economica* **26** (1959) 121–127.

[2] K.J. Arrow, *Social Choice and Individual Values* (Wiley, New York, 2nd ed., 1963).

[3] H. Kesten and N. Morse, A property of the multinomial distribution, *Annals of Mathematical Statistics* **30** (1959) 120–127.

[4] Y.J. Lee, Nonparametric selections in blocked data: application to motor vehicle fatality rate data, *Technometrics* **22** (1980) 535–542.

[5] Y.J. Lee and E.J. Dudewicz, Nonparametric ranking and selection procedures, Technical Report No. 105, Department of Statistics, The Ohio State University, Columbus, Ohio, 1974.

[6] Y.J. Lee and E.J. Dudewicz, Awardee selection procedures, with special reference to the Frank Wilcoxon and Jack Youden Prizes, *Technometrics* **22** (1980) 121–124.